Morse-Sard Theorem, Immersions and Embeddings

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Exercise 1.

1. Show that every immersed n-manifold (that is a manifold of dimension $n > 0$) of $\mathbb{R}^n$ is parallelizable.
2. Is it possible to immerse a compact n-manifold in $\mathbb{R}^n$?
3. What is the minimal number of charts an atlas of $\mathbb{S}^n$ can have?

Exercise 2 (Veronese embedding)**.** Recall that the projective space $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by equivalence relation "being in the same vector line". If $x = (x_0, \dots, x_n)$ is an element of $\mathbb{R}^{n+1} \setminus \{0\}$, we denote by $[x_0 : \dots : x_n]$ its projection in $\mathbb{RP}^n$. Let $h : \mathbb{RP}^2 \to \mathbb{RP}^5$ be the map defined by $h([x : y : z]) = [x^2 : y^2 : z^2 : xy : yz : zx]$.

1. Prove that h is well defined.
2. Prove that h is an embedding.

Exercise 3. Let M be a submanifold of $\mathbb{R}^n$ of dimension m with $2m < n$.

1. Show that for all $\varepsilon > 0$, there exists $v \in \mathbb{R}^n$ with $\|v\| < \varepsilon$ such that $(M + v) \cap M = \varnothing$.
2. *(Bonus)* What if $n \leqslant 2m$?

Exercise 4. Let M be a manifold, and V be a linear subspace of $\mathcal{C}^\infty(M)$ that contains the constant maps.

1. Prove that $\Sigma = \{(f, x) \in V \times M \mid f(x) = 0\}$ is a hypersurface of $V \times M$, and describe $T_{(f,x)}\Sigma$.
2. In this question, $M = \mathbb{R}$.
   (a) Let $(f, x) \in \Sigma$ such that $f'(x) \neq 0$. Show that there exists U and V, open neighborhoods of f and x and a smooth map $\varphi : U \to V$ such that $\varphi(f) = x$ and $g(\varphi(g)) = 0$ for all $g \in U$.
   (b) Deduce that the simple roots of a polynomial map in $\mathbb{R}_d[X]$ are smooth maps of the coefficients.
   (c) *(Bonus)* What happens for the multiple roots?
3. Let p_V and p_M be the projections from Σ to V and M.
   (a) Show that p_M is a submersion.
   (b) Find the critical points of p_V as well as its critical values.
   (c) Show that the set of $f \in V$ such that $f^{-1}(0)$ is a hypersurface of M has full measure.